HEAT CONDUCTION EQUATIONS ALLOWING FOR FINITE PHONON VELOCITY

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Using the impulse method the author obtained integral and differential forms of a nonlinear heat conduction equation with allowance made for the finiteness of the velocity of heat propagation for an arbitrarily shaped solid.

Introduction. The consistent development of the thermal impulse method [1], which had initially been called, on T. L. Perel'man's suggestion, the integral method of zero-order heat moments in formulating and solving problems of heat conduction [2, 3, 4], led to its establishment as a new law of the nonequilibrium thermodynamics of complex systems – an impulse principle (see [4-9]). Below it is shown how this principle is used to derive heat conduction equations in both integral and differential forms that allow for the finite velocity of phonon propagation in condensed solids. In [1-9], the author described already in sufficient detail not only the thermal impulse theory but also its thermodynamic extension with practical applications, particularly, in physical geocryology. Therefore, in the given article there is no need to present again proofs and detailed descriptions of the derivation of the basic relations. The aim of the work is not only to demonstrate the efficiency of the thermodynamic integral impulse method in a concretely formulated problem but also to show that the results of its solution by this method lead to a more complete description of the physical phenomenon.

In the vast majority of cases, allowance for the boundedness of the velocity w_s of phonons (sound) is unimportant, but in brief high-gradient thermal processes (heat "shocks") in thin surface layers of solids, for example, in solving problems of supersonic aerodynamics, it can be necessary. In the differential heat conduction (Fourier) equation, an additional term with w_s in the denominator should appear in this case, so that it drops out as $w_s \to \infty$. This approach to the description of transient heat conduction was proposed independently – by A. V. Luikov (1941) and then by P. Vernott (1961) and J. Tavernier (1962). There are already numerous publications to date, including monographs devoted to problems of heat conduction for a finite phonon velocity (see [10-13]).

The differential heat conduction equation appeared as [10]

$$\frac{\delta T}{d\tau} + \tau_t \frac{\delta^2 T}{d\tau^2} = a \frac{\partial^2 T}{\partial x^2}, \quad {}^{*)} \tag{1}$$

where $a = \lambda / c$; τ_t is the time constant of the transmission of thermal excitation by phonons or the relaxation time.

* Let us agree on the notation. While in classical thermodynamics, no allowance is made for time at all, and the local (for the entire volume of the working medium) change in temperature is always denoted by dT, in heat-conduction theory there is a need to distinguish the local change in the temperature with time and its change on displacement along x. Therefore, the partial derivatives $\partial T/\partial x$ and $\partial T/\partial \tau$ are used, and by ∂T an exact differential is meant. And what would be written as $dT/d\tau$ in classical thermodynamics will be $\partial T/\partial \tau$ in heat-conduction theory. To make the notation of heat-conduction theory approach that accepted in thermodynamics, let us denote the elementary change in temperature on displacement along x as $\partial T \equiv \partial T(x, \tau)$ and the elementary local increment in temperature with the time τ as δT in order not to confuse it with the exact differential dT. Therefore the partial derivative $\partial T/\partial \tau$ will be written as $\delta T/d\tau$.

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We note that τ_t is difficult to determine by experiment and it is given approximately as $\tau_t = a/w_s^2$ in monograph [10]. However, in [2, 4], it was shown that the time of temperature relaxation in classically shaped solids is approximately equal to $l^2/2af$ as Bi $\to \infty$ and, therefore, should be $\tau_t = 2af/w_s^2$. And this indicates that the accuracy of setting of the coefficient τ_t is very low when concrete problems are solved.

Owing to the presence of the second term on its left-hand side, Eq. (1) is hyperbolic. And we attemp to arrive at this eqution by using the integral regularity of the impulse – the momentum of energy, established earlier in [1, 4-6, 8, 9], assuming that this regularity will enable us to explicitly use the sound velocity w_s in the solid, which is determinable by experiment more accurately than τ_t.

Physical Principles in the Impulse Method. The basis of this method is formed by the idea of the substantiality of energy. It is interpreted not as a derivative of the category of motion (the relational concept) but quite the reverse – energy produces motion; it moves in space, too. Historically, the relational and substantial concepts of energy have alternated. The most decisive step in the development of the substantial concept belongs to N. A. Umov (1874) in his Doctor's Dissertation "Equations of Energy Motion in Solids." This concept, in fact, established the principle of energy motion in physics; mechanical work and heat as basic concepts in determining the first law of thermodynamics, in this case, become understood as two qualitatively different forms of the motion of energy through the boundary surface of the thermodynamic system under study. This principle was further developed in [1, 5, 6], where energy motion, for any of the degrees of freedom for the system, is meant not simply in a three-dimensional space but as overcoming resistance – time resistance (I). This resistance should also be interpreted substantially with the idea of its being distributed in the universe, like energy, everywhere in space and over the qualitative levels of matter. Owing to this, the notion of universal time resistance I or intergy (from the Latin "I place between") is introduced, which has, for different degrees of freedom of the system, qualitatively different values, for example, I_{therm} is thermal intergy, I_m is mechanical intergy, etc. The expression for the intergy (time resistance) of mechanical motion is special. For discrete particles and solids, it is numerically equal to the reciprocal of the velocity of their motion (see [8, 9]). And, thus, the energy and the intergy for any degrees of freedom of thermodynamic system always interact in pairs: working by the energy against the resistance, or the intergy I leads to the appearance of time for any process of energy dissipation. In [1], it was shown by examples from problems of physical geocryology that any complex thermodynamic system, in essence, forms a certain structure of intergies of individual degrees of freedom. And, fundamentally, the energy can move either for a parallel arrangement of the intergies of individual degrees of freedom or for their series arrangement. And these two possible cases are a analogs of Kirchhoff's laws for parallel circuits. We can see examples of the parallel arrangement of the intergies of individual degrees of freedom in a thermodynamic system in a classical formulation of the law of conservation of energy (the 1st law of thermodynamics) for any point of the control surface of the thermodynamic system. But it is precisely when allowance is made for the finiteness of the velocity of phonon propagation in the process of heat conduction in a solid that the example of series arrangement for the intergies of individual degrees of freedom of the thermodynamic system is revealed. This finite velocity, or, more precisely, its reciprocal, is, in fact, an addition to the thermal intergy I_{therm} that retards the process of heat conduction. We note that a great advantage in analyzing processes using the notion of intergy is that it always has the same dimensionality – the dimensionality of the reciprocal of the mechanical velocity of motion (sec/m) – for any degree of freedom of the thermodynamic system.

Comparison and Analysis of the Equations. For convenience, let us speak of the cooling of solids. Since phonons move inertially, as it were, along any curvilinear coordinate x of their flow, their velocity w_s will have an effect only purely on the transient portion of the moving internal heat energy (heat) in the solid, in other words – only on the zone of heat release σ in the solid in the coordinates $T-x$; the area of the region σ has the dimensionality (m·deg.) Then from the principle of time resistance (intergy), we can characterize the motion of the unsteady flow of internal heat energy (heat) in a solid as being performed through two series resistances or intergies thermal intergy $I_{therm} \equiv I = JN$ and mechanical (iterational) intergy $I_m = 1/|w_s|$; in other words, the total intergy is equal to:

$$I_{\text{total}} = I + I_{\text{m}} = JN + \frac{1}{|w_{\text{s}}|} \quad (\text{sec/m}), \tag{2}$$

where

$$I = I(T, x); \quad N = C(T, x)\,\omega(x);$$

$$J = J(T, x) = J_{\text{surf}} + J_{\text{int}} \equiv \frac{1}{\alpha(T, x)\,\omega(0)} + \int_0^x \frac{d\zeta}{\lambda(T, x, \zeta)\,\omega(\zeta)};$$

$\alpha(T, x)$ and $\lambda(T, x, \xi)$ are functions obtained by substituting into the coefficients as functions of temperature the profiles of the temperature distribution along x for any instant of the time τ [4]; $a = \lambda/c$ is the thermal diffusivity coefficient, m^2/sec.

According to the impulse method, we formulate the integral thermodynamic problem in the following manner. We determine the elementary momentum of the internal heat energy (heat) that corresponds to an elementary change in the area of the region σ:

$$d^2H = I_{\text{total}}\, d^2\sigma \equiv I_{\text{total}}\, \delta T(x, \tau)\, dx \equiv$$

$$\equiv \left(J(T, x)\, C(T, x)\, \omega(x) + \frac{1}{|w_{\text{s}}|} \right) \delta T(x, \tau)\, dx. \tag{3}$$

This quantity is always numerically equal to an elementary thermal impulse, which is also of the second order of smallness

$$d^2H = d^2\Omega \equiv \partial T d\tau. \tag{4}$$

But the total definiteness, i.e., only on the first-order of smallness, is obtained only after integrating expression (4) over the entire interval $[0, \hat{x}]$. And then this yields the desired elementary relation between the thermal impulse and the momentum of the internal heat energy (heat) that is fulfilled in an elementary time $d\tau$ of the unsteady-state thermal process:

$$d\Omega(\tau) \equiv (T_{\text{surf}}(\tau) - \hat{T}(\tau))\, d\tau = \int_0^{\hat{x}} I_{\text{total}}(T, x)\, d^2\sigma \equiv \int_0^{\hat{x}} I_{\text{total}}(T, x)\, \delta T(x, \tau)\, dx, \tag{5}$$

where $T_{\text{surf}}(\tau) \equiv T(0,\tau)$; $\hat{T}(\tau) \equiv T(\hat{x},\tau)$.

If, in this case, a phase transition occurs in the solid, we are able to convert the heat of the phase transition Q_{ph} to the equivalent heat capacity $C_{\text{equ}} = Q_{\text{ph}}/|T_{\text{ph}} - T_{\text{med}}|$, which is referred to the entire interval $T_{\text{ph}} - T_{\text{med}}$, and the total heat capacity on it will be $C = C_{\text{m}} + C_{\text{equ}}$, while the general form of formulation equation (5) is preserved.

For an arbitrary point x inside the solid, $0 \le x \le \hat{x}$; relation (5) will be written as:

$$(T(x, \tau) - \hat{T}(\tau))\, d\tau = \int_x^{\hat{x}} I_{\text{total}}(T, x, \zeta)\, \delta T(\zeta, \tau)\, d\zeta \tag{6}$$

or in view of expression (2)

$$(T(x, \tau) - \hat{T}(\tau))\, d\tau = \int_x^{\hat{x}} I(T, x, \zeta)\, \delta T(\zeta, \tau)\, d\zeta + \frac{1}{|w_{\text{s}}|} \int_x^{\hat{x}} \delta T(\zeta, \tau)\, d\zeta. \tag{7}$$

Replacing in (7) the integral with I by another form of iterated integral (see [4]), we have:

$$T(x,\tau) - \hat{T}(\tau) = \int_x^{\hat{x}} \frac{d\eta}{\lambda(T,\eta)\,\omega(\eta)} \int_\eta^{\hat{x}} C(T,\zeta) \frac{\delta T(\zeta,\tau)}{d\tau}\,\omega(\zeta)\,d\zeta +$$

$$+ \frac{1}{|w_s|} \int_x^{\hat{x}} \frac{\delta T(\zeta,\tau)}{d\tau}\,d\zeta\,. \tag{8}$$

Let us differentiate (8) to obtain again from the given integral form a differential heat conduction equation of a hyperbolic type. Upon the first differentiation with respect to x we obtain

$$\frac{\partial T(x,\tau)}{\partial x} = -\frac{1}{\lambda(T,x)\,\omega(x)} \int_x^{\hat{x}} C(T,\zeta)\,\omega(\zeta) \frac{\partial T(\zeta,\tau)}{d\tau}\,d\zeta - \frac{1}{|w_s|} \frac{\delta T(x,\tau)}{d\tau}\,,$$

or, allowing for the directivity of the vector of the velocity w_s:

$$\lambda(T,x)\,\omega(x) \frac{\partial T(x,\tau)}{\partial x} = -\int_x^{\hat{x}} C(T,\zeta)\,\omega(\zeta) \frac{\delta T(\zeta,\tau)}{d\tau}\,d\zeta +$$

$$+ \frac{\lambda(T,x)\,\omega(x)}{w_s} \frac{\delta T(x,\tau)}{d\tau}\,. \tag{9}$$

Differentiation of (9) once again with respect to x yields

$$\frac{\partial}{\partial x}\left(\lambda(T,x)\,\omega(x) \frac{\partial T}{\partial x}\right) = C(T,x)\,\omega(x) \frac{\delta T}{d\tau} +$$

$$+ \frac{\lambda(T,x)\,\omega(x)}{w_s} \frac{\delta^2 T}{dx d\tau} + \frac{1}{w_s} \frac{\delta T}{d\tau} \frac{\partial}{\partial x}(\lambda(T,x)\,\omega(x))\,,$$

or finally

$$\lambda(T,x)\,\omega(x) \frac{\partial^2 T}{\partial x^2} + \left(\frac{\partial T}{\partial x} - \frac{1}{w_s} \frac{\delta T}{d\tau}\right) \frac{\partial}{\partial x}(\lambda(T,x)\,\omega(x)) =$$

$$= C(T,x)\,\omega(x) \frac{\delta T}{d\tau} + \frac{\lambda(T,x)\,\omega(x)}{w_s} \frac{\delta^2 T}{dx d\tau}\,. \tag{10}$$

This is a nonlinear hyperbolic heat conduction equation but with a mixed derivative, unlike (1).

For constant λ and C, Eq. (10) appears as

$$\frac{1}{a} \frac{\delta T}{d\tau} + \frac{1}{w_s} \frac{\delta^2 T}{dx d\tau} = \frac{\partial^2 T}{\partial x^2} + \left(\frac{\partial T}{\partial x} - \frac{1}{w_s} \frac{\delta T}{d\tau}\right) K_x(x)\,, \tag{11}$$

where $K_x(x) = \omega'(x)/\omega(x) \equiv R'_{1x}(x)/R_{1x}(x) + R'_{2x}(x)/R_{2x}(x)$ is the overall ("average") curvature of the isothermal surface at the point x; for example, for $x = 0$ on the solid surface $K_x = 1/(l - x)$ for an infinite cylinder; $K_x = 2/(l - x)$ for a sphere; $K_x = 0$ for a plane solid.

Expression (11) corresponds in the literature to A. D. Chernyshov's heat conduction equation, which for some reason, does not contain the term $(1/w_s)(\delta T/\delta t)$. For the plane case, $\omega(x) = 1$, Eq. (11) is even more simplified:

$$\frac{\delta T}{d\tau} + \frac{a}{w_s} \frac{\delta^2 T}{dx d\tau} = a \frac{\partial^2 T}{\partial x^2}\,. \tag{12}$$

We note that, formally, the Chernyshov equation (12) becomes easily the most widely established Luikov–Vernott equation (1) for $\tau_t = a/w_s^2$ if the combined derivative is rewritten $\delta^2 T/dx d\tau = (\delta^2 T/d\tau^2)/(d\tau/dx)$ and by convention $dx/d\tau = w_s$. Then it is found that the inexactness of Eq. (1) should disappear when the velocities of the motion of isothermal surfaces in the solid approach the sound velocity in it, and these are cases of very thin films.

Thus, the problem of forming differential heat conduction equations with allowance for the finiteness of the velocity of phonon propagation turned out to be very useful for analysis in the context of the thermodynamic impulse method and those physical principles that are built in it. In this problem, the regularity of the additivity of thermal and mechanical intergies that acted in the given physical process not only as physical quantities of the same dimensionality but also of the same nature manifested itself clearly. Where the same form of interaction occurs (in the given case, this is a thermal process) and energy motion has simultaneously both thermal (relaxation) and mechanical forms, their intergies are summed, signifying by this that they are a series of resistances (of a special kind – time resistances) in the path of energy flow. It is precisely by this method that we can allow for the finiteness of the velocity of propagation of any form of interaction.

Equation (5) is an integral form of the hyperbolic heat conduction equation. Owing to the additivity of the intergies I and I_m we can identify a term, that integrally allows for the finiteness of the velocity of phonons (sound) that is an addition to the total thermal impulse in the entire time of the process τ:

$$\Delta H_s \equiv \int\int \frac{1}{w_s} \delta T dx \equiv \frac{1}{w_s} \int\int d^2\sigma = \frac{\sigma}{w_s}.$$

The same equation can also be written in integro-differential form – in the time interval $d\tau$

$$\delta H_s \equiv \frac{1}{w_s} \int_0^{\hat{x}} \delta T (x, \tau)\, dx$$

to allow for the additional momentum of energy per instant of time with subsequent superposition of the solutions in numerical calculations.

Conclusion. Thus, in the context of the fundamental physical principles of the thermodynamic impulse theory, we have interpreted the known heat conduction equations that allow for the finiteness of the velocity of phonon propagation. It is shown that these equations are somewhat inexact – in both their structure and coefficients. A more refined form of the nonlinear differential heat conduction equation (10) of a hyperbolic type with a mixed derivative for an arbitrarily shaped solid is obtained that includes plane, cylindrical, and spherical symmetries. This equation differs from the known analogous Chernyshov heat conduction equation by the presence of the term $(1/w_s)(\delta T/d\tau)$. Admittedly, the Chernyshov equation in its physical essence is more exact than the Luikov–Vernott equation only if the velocity of motion for isothermal surfaces in the solid approaches the velocity w_s of phonous (sound). Therefore, apparently, the Luikov–Vernott equation works well in fast processes only in very thin films. If we must allow for the finiteness of phonon velocity in thicker solids, we should use either the Chernyshov equation or the even more refined equation (10).

NOTATION

τ, time, sec; x, curvilinear space coordinate along the heat-flux trajectory in a solid for any instant of time τ and simultaneously the curvilinear axis of the heat-flux tube, chosen by us, which in the general case along this x can contract or expand; $w(x)$, area of curvilinear isothermal surface, variable along x and characterized by two main normal curvatures $K_{1x} = R'_{1x}/R_{1x}$ and $K_{2x} = R'_{2x}/R_{2x}$; R_{1x} and R_{2x}, two principal normal radii of curvature for isothermal surface at point x; the derivatives R'_{1x} and R'_{2x} yield only the sign of the curvatures (see [8, 9]), i.e., are equal either to $+1$ or to (-1); $\omega(x) = R_{1x}R_{2x}\varphi_{solid}$, φ_{solid}, solid inclusion angle, for example, $\omega(x) = 1$ for a plane section of surface; $\omega(x) = R_{1x}\varphi$ for a cylindrical section of surface with radius of curvature R_{1x} and inclusion angle φ; the angles φ and φ_{solid} can be chosen arbitrarily (for the maximum $\varphi = 2\pi$ and $\varphi_{solid} = 4\pi$); $x = 0$, on the solid surface; $x = \hat{x}$, point of thermal perturbation in a solid, most distant from the solid surface; if the solid is of finite dimensions, $\hat{x} = l$ is its characteristic dimension, for example, the half-thickness for a plate, the radius, for a cylinder or a sphere; λ, thermal conductivity coefficient, W/m·K; $C \equiv C_{eff}$, effective heat capacity per unit volume, dependent in the general case on x and T, kJ/m^3·K; T, temperature, K; T_{ph}, T_{surf}, and T_{med}, temperature of the phase transition, solid surface, and medium; Q_{ph}, heat of phase transition, kJ/m^3; σ, area of the region of heat release in time τ in the coordinates $T - x$ that takes place only in an unsteady process in a solid, m·K; α, heat-transfer coefficient on the solid surface, W/m^2·K, $Bi = \alpha l/\lambda$, Biot number; $J = J_{surf} + J_{int}$, thermal resistance along x from point $x = x^*$ to the medium, K/W; J_{surf}, thermal resistance on the solid surface; J_{int}, internal thermal resistance.

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